

# The Two-State Vector Formalism

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The two-state vector formalism (TSVF) [1] is a time-symmetric description of the standard quantum mechanics originated in Aharonov, Bergmann and Lebowitz [2]. The TSVF describes a quantum system at a particular time by two quantum states: the usual one, evolving forward in time, defined by the results of a complete measurement at the earlier time, and by the quantum state evolving backward in time, defined by the results of a complete measurement at a later time.

According to the standard quantum formalism, an ideal (von Neumann) measurement at time  $t$  of a nondegenerate variable  $A$  tests for existence at this time of the forward evolving state  $|A = a\rangle$  (it yields the outcome  $A = a$  with certainty if this was the state) and creates the state evolving towards the future:

$$|\Psi(t')\rangle = e^{-\frac{i}{\hbar} \int_t^{t'} H dt} |A = a\rangle, \quad t' > t. \quad (1)$$

(In general, the Hamiltonians  $H(t)$  at different times do not commute and a time ordering has to be performed.)

In the TSVF this ideal measurement also tests for backward evolving state arriving from the future  $\langle A = a|$  and creates the state evolving towards the past:

$$\langle \Phi(t'')| = \langle A = a| e^{\frac{i}{\hbar} \int_t^{t''} H dt}, \quad t'' < t. \quad (2)$$

Apart from some differences (discussed below) following from the asymmetry of the memory arrow of time, one can perform similar manipulations of the forward and backward evolving states. In particular, neither can be cloned and both can be teleported.

Given complete measurements,  $|A = a\rangle$  at  $t_1$  and  $|B = b\rangle$  at  $t_2$ , the complete description of a quantum system at time  $t$ ,  $t_1 < t < t_2$ , is the *two-state vector* [3]:

$$\langle\Phi| \quad |\Psi\rangle, \quad (3)$$

where the states  $\langle\Phi|$  and  $|\Psi\rangle$  are obtained using (1,2).

The two-state vector provides the maximal information regarding the way the quantum system can affect at time  $t$  any other system. In particular, the two-state vector describes the influence on a measuring device coupled with the system at time  $t$ . An ideal measurement of a variable  $O$  yields an eigenvalue  $o_n$  with probability given by the Aharonov, Bergman, Lebowitz (ABL) rule:

$$\text{Prob}(o_n) = \frac{|\langle\Phi|\mathbf{P}_{O=o_n}|\Psi\rangle|^2}{\sum_j |\langle\Phi|\mathbf{P}_{O=o_j}|\Psi\rangle|^2}. \quad (4)$$

This is, essentially, a conditional probability. We consider an ensemble of pre- and post-selected quantum systems with the desired outcomes of the measurements at  $t_1$  and  $t_2$ . Only those systems (and all of them) are taken into account. Intermediate measurement (or the absence of it) might change the probabilities of the outcomes of the post-selection measurement at time  $t_2$ , but this is irrelevant: it only changes the size of the pre- and post-selected ensemble given the size of the pre-elected ensemble at  $t_1$ .

Note that the ABL rule simplifies the calculation of probabilities of the outcome of intermediate measurements. In the standard approach we need to calculate the time evolutions between time  $t$  and  $t_2$  of all states corresponding to all possible outcomes of the intermediate measurement, while in the TSVF we have to calculate evolution of only one (backward evolving) state.

The pre- and post-selected quantum system (described by the two-state vector) has very different features relative to the system described by a single, forward evolving quantum state. The Heisenberg Principle does not hold: noncommuting observables might be simultaneously well defined, i.e. each observable might have a dispersion-free value provided that it was the only one measured at time  $t$ . As an example, consider a spin- $\frac{1}{2}$  particle in a field free region. Assume that  $\sigma_z$  was measured at  $t_1$ ,  $\sigma_x$  at  $t_2$  and both were found to be 1. When at time  $t$ , an outcome of a measurement of a variable (if measured) is known with certainty, it is named *an element of reality* [8]. Thus, in the above example, both  $\sigma_z = 1$  and  $\sigma_x = 1$  are such elements of reality.

For pre- and post-selected systems there might be apparently contradicting elements of reality. Consider now a spin- $\frac{1}{2}$  particle which can be located in two boxes,  $A$  and  $B$ , which is described by the two-state vector:

$$\langle\Phi| |\Psi\rangle = \frac{1}{3} (\langle A, \uparrow_z| + \langle A, \downarrow_z| - \langle B, \uparrow_z|) (|A, \uparrow_z\rangle + |A, \downarrow_z\rangle + |B, \uparrow_z\rangle), \quad (5)$$

(where  $|A, \uparrow_z\rangle$  represents the particle in box  $A$  with spin  $\uparrow_z$ ). Then, there are two elements of reality: “the particle in box  $A$  with spin up” and “the particle in box  $A$  with spin down”. Indeed, the measurement of the projection  $\mathbf{P}_{A\uparrow}$  has the outcome  $\mathbf{P}_{A\uparrow} = 1$  with certainty, and the outcome of the other projection (if measured instead) is also certain:  $\mathbf{P}_{A\downarrow} = 1$ . This can be readily verified using the ABL rule or the standard formalism.

Obviously, the measurement of the product of the projections is certain too:  $\mathbf{P}_{A\uparrow} \mathbf{P}_{A\downarrow} = 0$ , so this example shows also *the failure of the product rule*: at time  $t$  we know with certainty that if  $A$  is measured, the outcome is  $a$ , and if  $B$  is measured instead, the outcome is  $b$ , but nevertheless, the measurement of  $AB$  is not  $ab$ . (The product rule does hold for the standard, pre-selected quantum systems.)

This example is mathematically equivalent to the three-box paradox [4] in which a single pre- and post-selected particle can be found with certainty both in box  $A$  if searched there and in box  $B$  if searched there instead. These bizarre properties of elements of reality generated much controversy about the *counterfactual* usage of the ABL rule (see entry Counterfactuals in Quantum Mechanics). It should be stressed that “elements of reality” should not be understood in the ontological sense, but only in the operational sense, given by their definition.

The most important outcome of the TSVF is the discovery of *weak values* of physical variables [5]. When at time  $t$ , another system couples weakly to a variable  $O$  of a pre- and post-selected system  $\langle\Phi| |\Psi\rangle$ , the effective coupling is not to one of the eigenvalues, but to the weak value:

$$O_w \equiv \frac{\langle\Phi|O|\Psi\rangle}{\langle\Phi|\Psi\rangle}. \quad (6)$$

The weak value might be far away from the range of the eigenvalues, and this can lead to numerous surprising effects described in the entry “Weak Value and Weak Measurement”.

There is an important connection between weak and strong measurements. If the outcome of a strong measurement  $O = o_i$  is known with

certainty, the weak measurement has to yield the same value,  $O_w = o_i$ . The inverse is true for dichotomic variables: if the weak value is equal to one of the two eigenvalues, a strong measurement should give this outcome with certainty.

In both strong and weak measurements, the outcome manifests via the shift of the pointer variable. For strong measurements it might be random, but for weak measurements it is always certain (and equals to the weak value). Sometimes it is called “weak-measurement elements of reality” [9].

A generalization of the concept of the two-state vector (with natural generalizations of the ABL rule and weak value) is a “superposition” of two-state vectors which is called a *generalized two-state vector* [4]:

$$\sum_i \alpha_i \langle \Phi_i | \Psi_i \rangle. \quad (7)$$

A quantum system described by a generalized two-state vector requires pre- and post-selection of the system together with an ancilla which is not measured between the pre- and post-selection.

Systems described by generalized two-states vectors might have more unusual properties. The Heisenberg uncertainty principle breaks down in even more dramatic way: we can have a set of many noncommuting observables having dispersion-free values and not just the trivial case of two, one observable defined by pre-selection and another by post-selection. An extensively analyzed example of this kind is “the mean king problem” [6, 7] in which we have to know all observables of the set of the noncommuting observables for all possible outcomes of the post-selection measurement.

Another natural multiple-time nonlocal generalization is to consider  $2N$ -state vector (or generalized  $2N$ -state vector) which provides a complete description of how a (composite) system can affect other systems coupled to it in  $N$  space-time points. Preparing and testing such  $2N$ -state vectors require multiple-time and nonlocal measurements. (Note that causality puts some constraints on such measurements [10].) An incomplete description in which we associate only one (forward or backward) evolving state with some space-type points is also of interest. For example, two spin- $\frac{1}{2}$  particles in an entangled “state” which evolves forward in time for one particle and backward for the other particle, can be completely correlated:

$$\frac{1}{\sqrt{2}} (|\uparrow\rangle_A \langle\uparrow|_B + |\downarrow\rangle_A \langle\downarrow|_B). \quad (8)$$

Here, the measurements of the spin in components in any direction yield the same result for both particles. There is no pre-selected quantum system with such property.

The TSVF is a time symmetric approach. However, there are some differences between forward and backward evolving quantum states: we can always create a particular forward evolving quantum state, say  $|A = a\rangle$ . We measure  $A$ , and if the outcome is a different eigenvalue than  $a$ , we perform an appropriate transformation to the desired state. We cannot, however, create with certainty a particular backward evolving quantum state, since the correction has to be performed before we know the outcome of the measurement. The difference follows from the time asymmetry of the memory arrow of time. This asymmetry is not manifest in the ABL rule and the weak value, because the outcome of measurement is the *shift* of the pointer during the measurement interaction and this is invariant under changing the direction of time evolution. The shift is between zero and the outcome of the measurement and this is where the memory arrow of time introduces the asymmetry. The state “zero” is always in the earlier time: we do not “remember” the future and thus we cannot fix the final state of the measuring device to be zero.

The TSVF is equivalent to the standard quantum mechanics, but it is more convenient for analyzing the pre-and post-selected systems and allowed to see numerous surprising quantum effects. The TSVF is compatible with almost all interpretations of quantum mechanics but it fits particularly well the many-worlds interpretation. The concepts of “elements of reality” and “weak-measurement elements of reality” obtain a clear meaning in worlds with particular post-selection, while they have no ontological meaning in the scope of physical universe which incorporates all the worlds.

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